# Ultimate lateral pressure of two side-by-side piles in clay

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The ultimate earth resistance for a group of two side-by-side piles that are laterally loaded in clay is investigated using four different methods of analysis: three numerical (the displacement finite-element method, and the upper- and lower-bound finite-element limit analysis methods) and one analytical (an analytical upper-bound plasticity method developed in this paper). The results of the three numerical methods are shown to be in excellent agreement, while the analytical solution presents a theoretical upper bound that is very close to the numerical results. The results of the analyses are used to identify the predominant failure mechanisms for different pile spacings and pile–soil adhesions. They are also used to develop a design chart and design equations for determination of the ultimate lateral bearing capacity factor.

KEYWORDS: bearing capacity; clays; finite-element modelling; limit state design/analysis; piles; plasticity; theoretical analysis

### **INTRODUCTION**

The lateral resistance of piles in clay is commonly expressed in terms of the lateral bearing capacity factor,  $N_p$ , defined as

$$N_{\rm p} = \frac{p_{\rm u}}{s_{\rm u} D} \tag{1}$$

where  $p_u$  is the ultimate load per unit length,  $s_u$  is the undrained shear strength, and *D* is the pile diameter.  $N_p$ increases with depth from an initial low value at the ground surface to a maximum value at a certain depth, which corresponds to plane-strain soil movement around the pile, and remains constant thereafter. Various expressions have been proposed and are currently used in practice for the determination of  $N_p$  with depth for single piles in clay. These have been based on pile load tests (e.g. Matlock, 1970; Reese & Welch, 1975; Stevens & Audibert, 1980), model pile tests (e.g. Pan *et al.*, 2000; Jeanjean, 2009), three-dimensional displacement finite-element analysis (e.g. Brown & Shie, 1991; Yang & Jeremic, 2002; Georgiadis & Georgiadis, 2010) and analytical upper-bound limit analysis (Murff & Hamilton, 1993).

In the idealised framework of rigid-plastic limit analysis, the ultimate lateral bearing capacity factor (at large depths) for single piles depends exclusively on the pile-soil adhesion. Randolph & Houlsby (1984) presented two-dimensional lower and upper-bound plasticity calculations for the ultimate bearing capacity factor, which provided the exact solution for fully rough piles, and, following correction of an error pointed out by Murff *et al.* (1989), a good approximation for lower pile-soil adhesions. An improved upper-bound solution for low pile-soil adhesion and perfectly smooth piles was proposed by Christensen & Niewald (1992), while Martin & Randolph (2006) presented a combined upper-bound mechanism, which provides the best solution to date (exact for all practical purposes) for the whole range of pile-soil adhesions.

While the ultimate lateral bearing capacity of single piles

in clay has been established, very limited work has been published for the case of pile groups. It is common practice to take account of pile effects on lateral resistance through the application of a multiplication factor, called the *p*-multiplier, on the single-pile lateral load distribution. Such multiplication factors have been proposed by several investigators (Brown et al., 1988; Reese et al., 1996; Reese & Van Impe, 2001; Ilyas et al., 2004; Rollins et al., 2006), based mainly on full-scale pile load tests, and are included in design codes such as AASHTO (2007) and the US Army (1993). Owing to the high cost of fully instrumented lateral pile load tests, the available *p*-multipliers generally lack sufficient validation, and exhibit significant scatter. More importantly, since the proposed multipliers are back-calculated from measured pile-head loaddisplacement relationships, it is not possible to determine the variation with depth, and consequently a single value is assigned to each pile.

This paper presents an investigation of the ultimate lateral soil resistance (at large depths) on two piles that are laterally loaded parallel to their plane of symmetry, as shown in Fig. 1. Four methods of analysis are employed: the displacement finite-element method, the upper- and lower-bound finiteelement limit analysis methods, and the analytical upperbound plasticity method.

## FINITE-ELEMENT ANALYSES

A series of displacement finite-element analyses was performed with the computer program Plaxis 2D 2010 (Brinkgreve *et al.*, 2011) in which two infinitely long cylindrical



Fig. 1. Problem definition

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piles in an infinite elastic-perfectly plastic soil medium were displaced parallel to each other, as shown in Fig. 1. The piles were modelled as linear elastic with the elastic properties of reinforced concrete (Young's modulus  $E_p = 2.9 \times 10^7$  kPa and Poisson's ratio  $\nu_p = 0.1$ , while the soil was modelled as a Tresca material with undrained shear strength  $s_u = 100$  kPa, undrained Young's modulus  $E_u = 200s_u$  and undrained Poisson's ratio  $\nu_u = 0.495$ . Various adhesion factors  $\alpha$  ( $= \tau_f/s_u$ , where  $\tau_f$  is the limiting pile–soil adhesion) and pile spacings (s) were considered, while the pile diameter was taken as D = 1 m in all analyses.

Owing to the symmetry of the geometry and the loading conditions, only half of the problem was analysed. A typical finite-element mesh is shown in Fig. 2, for s /D = 3. It consists of approximately 7300 15-noded triangular elements for both the piles and the surrounding soil, with interface elements placed between the piles and the soil. In order to approximate rigid-plastic interface behaviour, high elastic normal and shear stiffnesses were assigned to the interface elements  $(11 \times 10^6 \text{ kN/m}^3 \text{ and } 10^6 \text{ kN/m}^3 \text{ respectively})$ . The same mesh density in the region around the pile was used in all analyses, while different pile spacings were modelled simply by translating the left mesh boundary towards or away from the pile. The other three boundaries were placed at a distance of 15*D* from the pile. All four boundaries of the mesh were fixed in the normal direction.

According to the Tresca failure criterion, which was adopted for the soil, the shear strength is independent of the normal stress level. Consequently, the finite-element results are independent of the initial stresses, and therefore these were specified as zero at the beginning of each analysis. Pile loading was subsequently simulated by applying prescribed displacement to all nodes on the pile diameter. The undrained bearing capacity factor  $N_p$  for each case was then calculated as the ratio of the resulting reaction force over  $s_u D$ . For the case of a single pile (i.e. with the left boundary sufficiently far away from the pile so that the failure mechanism and the resulting ultimate load are not affected), bearing capacity factors  $N_p$  of 9.19 and 11.95 were calculated for adhesion factors  $\alpha$  of 0 and 1 respectively. These values compare excellently with the lower bound solution of  $N_p = 9.14$  and the exact solution of  $N_p = 11.94$  (0.53% and 0.08% difference respectively) by Randolph & Houlsby (1984).

Typical curves of normalised load  $(p/s_uD)$  against normalised displacement (y/D) obtained from the finite-element analyses are shown in Fig. 3 for s/D = 3 and several adhesion factors ( $\alpha = 0.01$ , 0.25, 0.5, 0.75 and 1). As seen in this figure, a decrease in the adhesion factor leads to the expected lower calculated bearing capacity. It is also evident



Fig. 3. Curves of normalised load against normalised displacement for s/D = 3;  $\alpha = 0.01$ , 0.25, 0.5, 0.75 and 1



Fig. 2. (a) Finite-element mesh with boundaries; (b) detail of finite-element mesh around pile

in Fig. 3 that collapse is well defined for all adhesion factors considered, since a horizontal section of constant collapse load can be observed in all load-displacement curves.

The variation of the bearing capacity factor  $N_p (= p_u/s_uD)$ with the normalised pile spacing s/D is shown in Fig. 4 for the case of a fully rough pile ( $\alpha = 1$ ). As the pile spacing decreases, the bearing capacity factor  $N_p$  initially decreases from its single-pile value of 11.95, reaches a minimum value of 11.31 at s/D = 2.7, then increases to a peak value of 12.91 at s/D = 1.23, and finally drops sharply to 11.69 for the extreme case of two piles in contact with each other. This behaviour, and especially the peak of the bearing capacity factor at small pile spacing, is rather unexpected, and in contrast to the common assumption that the lateral



Fig. 4. Variation of  $N_p$  with s/D for rough pile ( $\alpha = 1$ )

bearing capacity should continuously reduce with a reduction in pile spacing. Such a continuous reduction is assumed, for example, in the *p*-multiplier method (e.g. Reese & Van Impe, 2001), although, as mentioned in the introduction, *p*-multipliers are generally obtained from pile-head load measurements, and therefore reflect the overall pile load reduction rather than the variation of the ultimate pressure that is examined in this study.

The variation of  $N_p$  with s/D illustrated in Fig. 4 can be explained by considering the different failure mechanisms that develop for different pile spacings. Fig. 5 shows the incremental displacements at failure for four normalised spacings s/D = 1.2, 1.5, 3 and 5. As seen in Fig. 5(a), when the piles are closely spaced (s/D = 1.2), the soil between them moves as a rigid body, and at the same velocity as the piles. In this case the two-pile group behaves as an equivalent non-circular pile of width s + D, resulting in an ultimate load for each pile of  $p_u = s_u N_{p,eq}(s + D)/2$ . According to this, the bearing capacity factor of each pile (equation (1)) becomes

$$N_{\rm p} = N_{\rm p,eq} \left( 1 + \frac{s/D - 1}{2} \right) \tag{2}$$

where  $N_{p,eq}$  is the undrained bearing capacity factor for the equivalent non-circular pile. As seen in Fig. 4, for s/D = 1,  $N_p$  is slightly lower than the single-pile value (11.69 compared with 11.95), indicating that the bearing capacity factor for the equivalent non-circular pile ( $N_{p,eq}$ ) is less than that of a circular pile. Substituting  $N_{p,eq} = 11.69$  into equation (2) provides an excellent approximation of  $N_p$  for failure mode B (Fig. 4), less than 1% higher than the finite-element



Fig. 5. Failure mechanisms from displacement finite-element analysis for rough pile ( $\alpha = 1$ ): (a) s/D = 1.2; (b) s/D = 1.5; (c) s/D = 3; (d) s/D = 5

analysis results. The analytical upper-bound solution presented below gives an even better estimation.

Beyond the pile spacing of s/D = 1.23, which corresponds to the peak value of  $N_p$  shown in Fig. 4, the failure mechanism changes abruptly and becomes smaller in extent, with the soil between the piles moving in the opposite direction to the piles, and undergoing plastic deformation. This is demonstrated clearly in Fig. 5(b), which shows the failure mechanism for s/D = 1.5. At this normalised spacing,  $N_p$  is still greater than the single-pile value, but reduces rapidly as the pile spacing increases, and becomes lower than the single-pile value beyond s/D = 1.6. This can be attributed to the overlapping of the mechanisms of the two piles.

Displacement finite-element analyses were also performed for the case of a line of equally spaced piles. The results of these analyses showed that the reduction of the ultimate resistance, compared with the single-pile case, at intermediate pile spacings of 2.5D and 3D is greater than that for a two-pile group by approximately 85% and 70% respectively.

## FINITE-ELEMENT LIMIT ANALYSES

The upper- and lower-bound theorems of plasticity are powerful tools for predicting the stability of geotechnical problems, but can be very cumbersome to apply in practice. Finite-element formulations of these theorems, which have evolved markedly over the last two decades, provide a new and exciting means of applying them to complex engineering problems in a routine manner.

Formally, the lower-bound theorem states that any stress field that satisfies equilibrium, the stress boundary conditions, and the yield criterion will support a load that does not exceed the true collapse load. Such a stress field is said to be statically admissible, and is the quantity that must be found in a lower-bound calculation. The upper-bound theorem, in contrast, requires the determination of a kinematically admissible velocity field that satisfies the velocity boundary conditions and the plastic flow rule. For such a velocity field, an upper bound on the collapse load is found by equating the power expended by the external loads to the power dissipated internally by plastic deformation. Both limit theorems assume a perfectly plastic material with an associated flow rule, and ignore the effect of geometry changes.

Finite-element limit analysis is particularly powerful when upper- and lower-bound estimates are calculated in tandem, so that the true collapse load is bracketed from above and below. The difference between the two bounds then provides an exact measure of the discretisation error in the solution, and can be used to refine the meshes until a suitably accurate estimate of the collapse load is found. The formulations used in this investigation stem from the methods originally developed by Sloan (1988, 1989), but have evolved significantly over the past two decades to incorporate the major improvements described in Lyamin & Sloan (2002a, 2002b) and Krabbenhoft et al. (2005, 2007). Key features of the methods include the use of linear finite elements to model the stress/velocity fields, and collapsed solid elements at all inter-element boundaries to simulate stress/velocity discontinuities. The solutions from the lowerbound formulation yield statically admissible stress fields, while those from the upper-bound formulation furnish kinematically admissible velocity fields. This ensures that the solutions preserve the important bounding properties of the limit theorems.

Both formulations result in convex mathematical programs, which (considering the dual form of the upper-bound problem) can be cast in the form maximise  $\lambda$ 

subject to 
$$\mathbf{A}\boldsymbol{\sigma} = \boldsymbol{p}_0 + \lambda \boldsymbol{p}$$
 (3)  
 $f_i(\boldsymbol{\sigma}) \leq 0, \ i = \{1, \dots, N\}$ 

where  $\lambda$  is a load multiplier,  $\sigma$  is a vector of stress variables, **A** is a matrix of equality constraint coefficients,  $p_0$  and p are vectors of prescribed and optimisable forces respectively,  $f_i$ is the yield function for stress set *i*, and *N* is the number of stress nodes. The solutions to problem (3) can be found efficiently by using specialised optimisation solvers that are based on interior-point methods or second-order cone programming (e.g. Lyamin & Sloan, 2002a, 2002b; Krabbenhoft *et al.*, 2007).

Figure 6 shows the finite-element mesh, the power dissipation, and the horizontal and vertical velocity field plots obtained from the upper-bound finite-element limit analysis for s/D = 3 and a fully rough pile ( $\alpha = 1$ ). Comparing Figs 5(c) and 6, it can be seen that the two finite-element methods predict very similar failure mechanisms. The bearing capacity factor  $N_p$  obtained from the displacement finiteelement analysis ( $N_p = 11.38$ ) lies between the values obtained from the upper- and lower-bound finite-element limit analyses (LB:  $N_p = 11.33$  and UB:  $N_p = 11.41$ ). A similarly excellent comparison of the displacement finiteelement analysis and finite-element limit analysis results was observed for all pile spacings and adhesion factors, as discussed in a subsequent section of this paper.

# ANALYTICAL UPPER-BOUND SOLUTION

Two kinematic mechanisms will be presented in this section: the first mechanism gives the optimum solution for very small pile spacings (failure mode B in Fig. 4), and the second is optimal for greater pile spacings (failure mode A in Fig. 4). As both mechanisms are based on existing solutions for single piles, these solutions will be first presented in brief.

#### Existing solutions for single piles

The ultimate lateral pile capacity of a single pile in undrained clay was calculated analytically by Randolph & Houlsby (1984), using the upper- and lower-bound plasticity theorems. One quarter of the upper-bound kinematic mechanism used in these calculations is shown in Fig. 7(a). The loading direction coincides with the y-axis, which is therefore an axis of symmetry. The x-axis is also an axis of symmetry with respect to the geometry of the mechanism, and an axis of antisymmetry with respect to the velocity field. The mechanism consists of a rigid region (ABC), which moves with the pile, four velocity discontinuities (AC, AO, CDE and AGF), and two plastically deforming regions (CDEFGA and AGFO). Sections CD and AG of the velocity discontinuities are involutes of a circle of radius  $\lambda R$  (dashed line in Fig. 7(a)), where  $\lambda$  is equal to  $\cos(\arccos \alpha/2)$ , while DE and GF are circular arcs with centre O.

For adhesion factors close to unity (rough pile/soil interface), the above kinematic mechanism yields results that are practically identical to the lower-bound solution proposed in the same paper by Randolph & Houlsby (1984). In the limiting case  $\alpha = 1$ ,  $N_p = 11.94$  for both the upper- and lower-bound solutions, so this value is exact. As the adhesion factor decreases, the upper-bound solution (as corrected by Murff *et al.*, 1989) based on this mechanism deviates from the lower-bound solution. For  $\alpha = 0$  (a perfectly smooth pile) the difference is 9.1%.

An alternative mechanism, which gives better results for



Fig. 6. Failure mechanism from finite-element limit analysis for s/D = 3 and  $\alpha = 1.0$ : (a) FELA mesh; (b) power dissipation; (c) horizontal velocities,  $v_x$ ; (d) vertical velocities,  $v_y$ 

lower values of  $\alpha$ , was presented by Christensen & Niewald (1992). This mechanism is shown in Fig. 7(b), and consists of only two velocity discontinuities (AC and AB) and a rigid region (ABC) that rotates about point O. For  $\alpha = 0$  this mechanism gives an upper bound that is only 0.8% higher than the lower bound of Randolph & Houlsby (1984). This difference increases with increasing  $\alpha$ , and is 5.5% at  $\alpha = 1$ .

Martin & Randolph (2006) proposed a combination of the above two mechanisms that provides the best solution to date over the whole range of  $\alpha$  values. This combined mechanism is shown in Fig. 7(c), and forms the basis on which the kinematic mechanisms for a two-pile group presented below were developed. As seen in Fig. 7(c), a rigid rotating body (XHI) similar to that shown in Fig. 7(b) is introduced within the plastically deforming region (AGFO). The optimum size of XHI decreases with increasing pile–soil adhesion, and disappears for  $\alpha = 1$ . Only one geometrical parameter, the angle  $\beta$ , is required to define the mechanism. The optimum mechanism for different adhesion factors,  $\alpha$ , is determined by calculating the value of  $\beta$  that minimises the calculated bearing capacity factor.

#### Solution for two side-by-side piles

Two kinematic mechanisms are presented in this section. The first mechanism, shown in Fig. 8, covers many practical cases and corresponds to mechanism A of Fig. 4, which was identified in the displacement finite-element results. Geometrical fourfold symmetry is also retained in the case of the two-pile group, and therefore only a quarter of the mechanism is shown in Fig. 8. Similar to the single-pile case, the *x*-axis is an axis of symmetry with respect to the geometry of the mechanism, and an axis of antisymmetry with respect to the velocity field. Six geometrical parameters are required to define the mechanism: the angles  $\beta_1$  and  $\beta_2$  and the normalised radius  $\lambda$  of the evolute circle for the outer part of the mechanism, the angle of rotation  $\omega$ , and the angles  $\beta'_1$ and  $\beta'_2$  of the inner part of the mechanism. The normalised radius of the evolute circle for the inner part of the mechanism is a function of  $\lambda$ ,  $\omega$ ,  $\beta_1$  and  $\beta'_1$ . According to Fig. 8

$$YC = \frac{\lambda R}{\sin(\beta_1 + \omega)} = \frac{\lambda' R}{\sin(\beta'_1 - \omega)}$$
(4a)

and therefore

$$\lambda' = \lambda \frac{\sin(\beta_1' - \omega)}{\sin(\beta_1 + \omega)} \tag{4b}$$

The optimum mechanism, for a given adhesion factor  $\alpha$  and pile spacing *s*, is found by determining the combination of the six optimisation parameters that results in the minimum



Fig. 7. Kinematic mechanisms for single pile: (a) Randolph & Houlsby (1984); (b) Christensen & Niewald (1992); (c) Martin & Randolph (2006)

calculated load, and consequently in the minimum bearing capacity factor.

As seen in Fig. 8, the outer part of the mechanism (on the right side of the pile) is similar to the single-pile mechanism of Martin & Randolph (2006). The main difference is that the apex (C) of the rigid region in front of the pile is not necessarily positioned on the pile axis of symmetry, as in the single-pile case, but is generally positioned at an angle  $\omega$  to it. In addition, unlike the single-pile case, the optimum solution is not obtained for  $\beta_1 = \pi/4$  and  $\lambda = \cos(\arccos \alpha/2)$ , but at different values depending on both the adhesion factor  $\alpha$  and the pile spacing *s*.

The velocities within the two deforming regions (CDEFGA and AGFIH) are parallel to discontinuities CDE, AGF and HI (i.e. there is only a tangential component). Since the boundaries are also parallel to each other, it follows that, in order to satisfy the zero volumetric strain condition (undrained conditions), the velocity may vary only in the radial direction. The same obviously applies to the velocities within the rigid rotating body (HIX). The magnitude of the velocity and its variation within each region of the mechanism are controlled by the velocity jump at the pile/soil interface. Referring to Figs 9(a), 9(b) and 9(c), the velocities v and relative velocities  $\Delta v$  of the outer mechanism can be calculated from the following equations as functions of the pile velocity  $v_0$ .

Region HIX (Fig. 8) – discontinuity HX (Fig. 9(a)):

$$v = v_0 \frac{\sin \theta}{\sin(\theta + \psi)} = v_0 \frac{\cos \beta_2}{\lambda R} r \text{ and}$$

$$\Delta v = v_0 \frac{\sin \psi}{\sin(\theta + \psi)} = v_0 \frac{\cos \beta_2}{\lambda}$$
(5)

Region AGFIH (Fig. 8) - discontinuity AH (Fig. 9(b)):

$$v = v_0 \frac{\sin(\beta + \arccos \lambda)}{\lambda} \text{ and}$$

$$\Delta v = v_0 \frac{\cos \beta}{\lambda}$$
(6)

Region CDEFGA (Fig. 8) - discontinuity AC (Fig. 9(c)):

$$v = v_0 \sin \beta_1$$
 and  $\Delta v = v_0 \cos \beta_1$  (7)

The work calculations for the outer part of the mechanism are similar to those of Randolph & Houlsby (1984) and Murff *et al.* (1989), and are briefly presented in the Appendix.

The inner part of the mechanism of Fig. 8 (on the left



Fig. 8. Kinematic mechanism for two-pile group (mechanism A)



Fig. 9. Velocities at pile/soil discontiuities for outer section of kinematic mechanism A: (a) interface HX; (b) interface AH; (c) interface AC

side of the pile) also includes two deforming regions (C'D'E'F'G'A' and A'G'F'I'H') and a rigid rotating block (H'I'X') similar to the single-pile case. However, as seen in Fig. 8, these extend to a velocity discontinuity (X'I'F'E') rather than the *x*-axis. This discontinuity allows the change of velocity direction that is necessary in order to satisfy the zero horizontal displacement boundary condition at the plane of symmetry (*y*-axis). In order to satisfy compatibility conditions at the velocity discontinuity, the area below it is divided into three regions: a rigid block (E'F'F''E'') and two deforming regions (F'I'I''F'' and I'X'I''). Depending on the values of the optimisation parameters, the centre O' of the circular arcs D'E', G'F' and H'I' may be positioned outside the pile. In this case the velocity discontinuity becomes a straight line (O'I'F'E').

The velocities, v, in H'I'X', A'G'F'I'H' and C'D'E'F'G'A' are given by equations (5)–(7), in which  $\beta$ ,  $\beta_1$ ,  $\beta_2$ , and  $\lambda$  are substituted by  $\beta'$ ,  $\beta'_1$ ,  $\beta'_2$  and  $\lambda'$ . The velocities, v', in the remainder of the mechanism are controlled by the velocity jump at the discontinuity X'F'I'E' (or O'F'I'E', depending on the position of O'). Based on Fig. 10, which shows the relationship between the velocities v, v' and  $\Delta v$ , and using equations (5)–(7) to substitute for v, the following relationships are derived for the velocities,  $\nu'$ , in the area below (X'I'F'E'), and the relative velocities,  $\Delta v$ , at the discontinuity, as a function of the pile velocity,  $v_0$ .

Region E'F'F''E'' – discontinuity E'F':

$$v' = \frac{v}{\cos\xi_0} = v_0 \frac{\sin\beta_1'}{\cos\xi_0} \text{ and}$$
(8)

$$\Delta v = v \tan \xi_0 = v_0 \sin \beta_1' \tan \xi_0$$

Region F'I'I''F'' – discontinuity F'I':

$$v' = \frac{v}{\cos\xi_0} = v_0 \frac{\sin(\beta' + \arccos\lambda')}{\lambda'\cos\xi_0}$$
(9a)

and

$$\Delta v = v \tan \xi_0 = v_0 \frac{\sin(\beta' + \arccos \lambda') \tan \xi_0}{\lambda'}$$
(9b)

Region I'X'I'' – discontinuity (I'X'):



Fig. 10. Velocities at and below discontinuity  $E^\prime F^\prime I^\prime X^\prime$  for kinematic mechanism A

$$v' = v \frac{\cos(\xi_1 - \xi)}{\cos \xi_1}$$
  
=  $\frac{v}{O'I'} \left( O'X' + \frac{d}{\cos \xi_1} \right)$   
=  $v_0 \left[ \frac{\cos \beta_2'}{\lambda' R} \left( R + \frac{d}{\cos \xi_1} \right) - 1 \right]$  (10a)

and

$$\Delta v = v \frac{\sin \xi}{\cos \xi_1} = v_0 \frac{d \cos \beta_2' \tan \xi_1}{\lambda' R}$$
(10b)

The velocity distribution determined through equations (8)–(10) is also shown in Fig. 10. The angles  $\xi_0$  and  $\xi_1$  of the mechanism are determined from the geometry of Fig. 10.

$$\xi_0 = \arccos\left[\frac{(s/2)/(\lambda'R) - \sec\beta'_2}{\beta'_1 - \beta'_2 + \tan\beta'_2 + \cot(\beta'_1 - \omega)}\right]$$
(11)

$$\xi_1 = \arctan\left[\frac{\tan\xi_0}{1 + (\lambda' - \cos\beta'_2)/\sin(\beta'_2 + \arccos\lambda')}\right] \quad (12)$$

The work calculations, which are based on the velocity field derived above for the inner section of the mechanism, are presented in the Appendix. Noting that the force acting on each pile is the only external force, the undrained bearing capacity factor for a given set of optimisation parameters is calculated through the equation

$$N_{\rm p} = \frac{\Delta W_{\rm p}/v_0}{s_{\rm u}D} \tag{13}$$

where  $\Delta W_p$  is the total work done by the stresses in half of the symmetric problem corresponding to one pile of the two-pile group. The optimum value of  $N_p$  for a given pile spacing and adhesion factor was calculated by determining the combination of optimisation parameters resulting in the minimum  $N_p$ .

As discussed in the previous section, a single failure mechanism (mechanism B in Fig. 4) that includes both piles becomes predominant in the special case of very closely spaced piles. Such an upper-bound kinematic mechanism, which is essentially a simple variation of the Martin & Randolph (2006) mechanism for single piles, is shown in Fig. 11. The rigid soil region (shaded area in Fig. 11) is much larger than that for a single pile (Fig. 7(c)), and includes the soil between the two piles. Only one geometric parameter, the angle  $\beta$ , is required to define the mechanism. Work calculations and the derivation of the bearing capacity factor  $N_{\rm p}$  are presented in the Appendix.

#### **RESULTS AND DESIGN EQUATIONS**

The displacement finite-element analysis, finite-element limit analysis and analytical upper-bound results are compared in Figs 12–14 for three values of the adhesion factor,  $\alpha = 0$ , 0.5 and 1. Excellent agreement is observed between the displacement finite-element analysis and finite-element limit analysis results for all adhesion factors The difference between the numerical upper and lower bounds is very small in all cases, ranging from 0.6% for  $\alpha = 1$  and s/D = 1.15 to 1.5% for  $\alpha = 0$  and s/D = 1.75. It is observed that the displacement finite-element results always fall within the numerical lower and upper bounds.

It can also be seen in Figs 12–14 that the analytical upperbound results are also in very good agreement with the numerical results. This agreement is excellent (< 1%) at small pile spacings (mechanism B), while at greater pile spacings the best agreement is observed for the case of a fully rough pile,  $\alpha = 1$  (Fig. 12), for which the upper-bound  $N_p$ 



Fig. 11. Kinematic mechanism for two-pile group (mechanism B)



Fig. 12. Comparison of finite-element analysis (FEA), finiteelement limit analysis (FELA) and analytical upper-bound (UB) results for  $\alpha = 1$ 



Fig. 13. Comparison of finite-element analysis (FEA), finiteelement limit analysis (FELA) and analytical upper-bound (UB) results for  $\alpha = 0.5$ 



Fig. 14. Comparison of finite-element analysis (FEA), finiteelement limit analysis (FELA) and analytical upper-bound (UB) results for  $\alpha = 0$ 

values are up to 3.6% higher than those from the displacement finite-element analysis. The difference between the analytical upper-bound and displacement finite-element results increases slightly as  $\alpha$  decreases, but generally remains less than 5%.

Further comparison of the solutions presented in this paper is shown in Figs 15 and 16. Fig. 15 presents the finite element incremental displacements at failure together with



Fig. 15. Incremental displacements at failure (FEA) and upperbound mechanism for  $\alpha = 0.5$  and s/D = 1.1



Fig. 16. Incremental displacements at failure (FEA) and upperbound mechanism for  $\alpha = 0.5$  and s/D = 2.5

the optimum analytical upper-bound mechanism for  $\alpha = 0.5$ and s/D = 1.1. The excellent agreement in the calculated  $N_p$ values, discussed in the previous paragraph, can also be observed in the predicted failure mechanism. This excellent agreement is observed for all adhesion factors and pile spacings for this type of failure (mechanism B of Fig. 4). For greater pile spacings the agreement among the different solutions is also very good, as seen in Fig. 16 for s/D = 2.5and  $\alpha = 0.5$ .

As discussed in the previous paragraphs, the displacement finite-element analysis results always lie between the closely spaced finite-element upper- and lower-bound numerical results, while the analytical solution presents a theoretical upper bound that is very close to the numerical results. It is therefore reasonable to consider the displacement finite-element results as the exact solution for all practical purposes.

The relationship between  $N_p$  and s/D for  $\alpha = 0, 0.25, 0.5, 0.75$  and 1, obtained from the displacement finite-element results, is shown in Fig. 17. As indicated, the relationship is similar for all values of  $\alpha$ . Two critical pile spacings can be identified for each value of  $\alpha$ : the spacing  $s_p/D$ , which corresponds to the peak bearing capacity factor  $N_{pp}$ ; and the spacing  $s_1/D$ , which corresponds to the single-pile bearing



Fig. 17. Variation of  $N_p$  with s/D for  $\alpha = 0$ , 0.25, 0.5, 0.75 and 1

capacity factor  $N_{p1}$  (Fig. 18). According to Fig. 17, both  $s_p/D$  and  $s_1/D$  vary with the adhesion factor  $\alpha$ , and can be approximated by the following linear equations.

$$s_{\rm p}/D = 1.05 + 0.18\alpha \tag{14}$$

$$s_1/D = 3 \cdot 1 + 1 \cdot 4\alpha \tag{15}$$

The bearing capacity factors  $N_{\rm p0}$  for s/D = 1 and  $N_{\rm pp}$  at peak can be expressed by

$$N_{\rm p0} = 10.35 + 1.4\alpha \tag{16}$$

$$N_{\rm pp} = N_{\rm p0} + 5.4 \left(\frac{s_{\rm p}}{D} - 1\right) \tag{17}$$

while the single-pile bearing capacity factor  $N_{p1}$  is given by the lower-bound analytical expression by Randolph & Houlsby (1984),

$$N_{\rm p1} = \pi + 2 \arcsin \alpha + 2 \cos(\arcsin \alpha) + 4 \left[ \cos\left(\frac{\arcsin \alpha}{2}\right) + \sin\left(\frac{\arcsin \alpha}{2}\right) \right]$$
(18)

For pile spacings between 1 and  $s_p/D$ ,  $N_p$  varies linearly between  $N_{p0}$  and  $N_{pp}$ . For pile spacings between  $s_p/D$  and  $s_1/D$ ,  $N_p$  varies nonlinearly, and can be accurately approximated by the expression

$$N_{\rm p} = N_{\rm pp} \left(\frac{s}{s_{\rm p}}\right)^b \left[\frac{N_{\rm p1}}{N_{\rm pp}} \left(\frac{s_{\rm p}}{s_{\rm 1}}\right)^b\right]^{\left[\left(s_{\rm p}/s\right)-1\right]/\left[\left(s_{\rm p}/s_{\rm 1}\right)-1\right]}$$
(19)

where  $b = 0.75/(1 + \alpha)$ .

The  $N_p$ -s/D relationships for various adhesion factors  $\alpha$ , obtained using equations (14)–(19), are compared with the numerical results in Fig. 17.



Fig. 18. Definitions for empirical  $N_p - s/D$  relationship

#### CONCLUSIONS

Four methods of analysis were employed to calculate the ultimate lateral earth pressure for two side-by-side piles: the displacement finite-element method, the upper- and lowerbound finite-element methods and the analytical upper-bound plasticity method. Based on the results of the analyses performed, two distinct failure mechanisms were identified: a mechanism that is predominant at small pile spacings, in which both piles and the soil between them move as one rigid body; and a second mechanism in which two separate but overlapping mechanisms develop for each pile. The maximum value of the lateral bearing capacity factor for a given pile– soil adhesion is observed at the transition point between the two mechanisms, while the lowest bearing capacity is observed for the second mechanism at intermediate pile spacings.

The results of all four methods used were shown to be in very good agreement. The displacement finite-element results always fall between the finite-element limit analysis bounds, while the new analytical solution gives a close theoretical upper bound to the numerical results.

Finally, a design chart and design equations were presented for the calculation of the ultimate lateral bearing capacity factor as a function of pile spacing and pile–soil adhesion.

APPENDIX

The dissipation along a velocity discontinuity is

$$\tau_{\rm f} \int_0^{L_i} v_i {\rm d}l \tag{20}$$

and within a plastically deforming region is

$$S_{u} \int_{A} \dot{\gamma} dA$$
 (21)

where  $v_i$  is the relative velocity and  $L_i$  is the length of discontinuity i,  $\dot{\gamma}$  is the shear strain rate, and  $\tau_f$  is the ultimate shear stress along the discontinuity (equal to  $s_u$  in the soil and  $\alpha s_u$  along the pile/soil interface). If  $v_i$  and  $\dot{\gamma}$  are negative over part of a discontinuity or plastically deforming region, respectively, then their absolute values need to be used in equations (20) and (21). This was not, however, the case in any of the calculations presented in this paper.

The shear strain rate in cylindrical coordinates  $(r, \theta, z)$  is given by the following equation (for  $v_z = 0$ ).

$$\dot{\gamma} = \frac{\nu_{\theta}}{r} - \frac{\partial \nu_{\theta}}{\partial r} - \frac{1}{r} \frac{\partial \nu_{r}}{\partial \theta}$$
(22)

Kinematic mechanism A

*Outer section mechanism.* The lengths of the velocity discontinuities of the partial mechanism of Fig. 19 are



Fig. 19. Outer upper-bound mechanism

$$\begin{split} L_{\rm HX} &= R(\beta_2 + \Lambda) \\ L_{\rm AH} &= R(\beta_1 - \beta_2) \\ L_{\rm CA} &= R \frac{\cos(\omega + \beta_1 + \Lambda)}{\sin(\omega + \beta_1)} \\ L_{\rm CD} &= \int_{\beta_2}^{\beta_1} r d\beta = \int_{\beta_2}^{\beta_1} \left\{ L_{\rm DG} + R[\sin\Lambda + \lambda(\beta_1 - \beta)] \right\} d\beta \\ &= R(\beta_1 - \beta_2) \left[ \frac{\cos(\omega + \beta_1 + \Lambda)}{\sin(\omega + \beta_1)} + \sin\Lambda + \frac{\lambda}{2}(\beta_1 - \beta_2) \right] \\ L_{\rm AG} &= R(\beta_1 - \beta_2) \left[ \sin\Lambda + \frac{\lambda(\beta_1 - \beta_2)}{2} \right] \\ L_{\rm GF} &= R \left( \frac{\pi}{2} + \beta_2 \right) [\lambda(\beta_1 - \beta_2 + \tan\beta_2) + \sin\Lambda] \\ L_{\rm DE} &= \left( \frac{\pi}{2} + \beta_2 \right) (L_{\rm CA} + L_{\rm GF}) \end{split}$$
(23)

where  $\Lambda = \arccos \lambda$ .

For undrained conditions the relative velocities are parallel to the velocity discontinuities. It is straightforward to show that if the pile has a velocity  $v_0$  parallel to the *y*-axis, the relative velocities for the partial mechanism are

$$v_{\rm HX} = v_0 \frac{\cos \beta_2}{\lambda}$$

$$v_{\rm AH} = v_0 \frac{\cos \beta}{\lambda}$$

$$v_{\rm CA} = v_0 \cos \beta_1$$

$$v_{\rm CD} = v_{\rm DE}$$

$$v_{\rm AG} = v_{\rm GF} = v_0 \left[ \frac{\sin(\beta_1 + \Lambda)}{\lambda} - \sin \beta_1 \right]$$
(24)

The work done along the velocity discontinuities for the outer part of the mechanism is

$$s_{u} \left[ \alpha \left( L_{HX} v_{HX} + v_{0} R \frac{\sin \beta_{1} - \sin \beta_{2}}{\lambda} \right) + L_{CA} v_{CA} + L_{CD} v_{CD} + L_{DE} v_{DE} + L_{AG} v_{AG} + L_{GF} v_{GF} \right]$$
(25)

The total work for this part of the mechanism includes the work done within the three plastically deforming regions AGH, CDEFGA and GFIH, as follows.

Region AGH  

$$s_{u}v_{0}R\int_{\beta_{2}+\Lambda}^{\beta_{1}+\Lambda}\int_{\beta_{2}+\Lambda}^{\beta+\Lambda}[\sin(\beta+\Lambda)-\cos(\beta+\Lambda)(\tan\Lambda+\beta-t)]dtd\beta$$

where the angles  $\beta$  and t are defined in Fig. 19. As discussed by Martin & Randolph (2006), in the Randolph & Houlsby (1984) mechanism, which includes a similar deforming region (AGO in Fig. 7(a)), the work integrand becomes negative in part of the region for  $\alpha < 1$ . This could be overcome by introducing the absolute value of the integrand in the above expression and performing the integration numerically. It was found, however, in this study that the introduction of the rigid rotating region HIX (Fig. 19) resolves this issue, and the above integrand is always positive in the optimum mechanism for any value of  $\alpha$ . The same applies to the similar integral of the inner section of the mechanism.

(26)

Integrating expression (26) gives the following work expression for (AGH).

$$v_{0}s_{u}R\left\{\begin{array}{l}-2\sin(\beta_{2}+\Lambda)+\cos(\beta_{2}+\Lambda)\frac{\sin\Lambda}{\lambda}\\-\cos(\beta_{1}+\Lambda)\left[2(\beta_{1}-\beta_{2})+\frac{\sin\Lambda}{\lambda}\right]\\-\frac{1}{2}\sin(\beta_{1}+\Lambda)\\\times\left[(\beta_{1}-\beta_{2})^{2}+2(\beta_{1}-\beta_{2})\frac{\sin\Lambda}{\lambda}-4\right]\end{array}\right\}$$
(27)

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where  $\Lambda' = \arccos \lambda'$ .

Region CDEFGA

$$v_0 s_u R \sin \beta_1 \int_0^{\beta_1 + \pi/2} \frac{\cos(\omega + \beta_1 + \Lambda)}{\sin(\omega + \beta_1)} d\chi$$

Therefore

$$v_0 s_u R \cos(\omega + \beta_1 + \Lambda) \frac{\sin(\beta_1)}{\sin(\omega + \beta_1)} \left(\frac{\pi}{2} + \beta_1\right)$$
(28)

Region GFIH

$$\begin{split} v_0 s_{\mathrm{u}} R \int_{\beta_2 + \Lambda}^{\beta_1 + \Lambda} \int_0^{(\pi/2) + \beta_2} [\sin(\beta + \Lambda) - \cos(\beta + \Lambda) \\ \times [\tan(\beta_2) + \tan\Lambda + \beta - \beta_2]] \mathrm{d}\chi \mathrm{d}\beta \end{split}$$

Therefore

$$v_{0}s_{u}R\left(\frac{\pi}{2}+\beta_{2}\right)\left\{\begin{array}{l}2\left[\cos(\beta_{2}+\Lambda)-\cos(\beta_{1}+\Lambda)\right]\\-\sin(\beta_{1}+\Lambda)\left[\beta_{1}-\beta_{2}+\tan\beta_{2}+\frac{\sin\Lambda}{\lambda}\right]\\+\sin(\beta_{2}+\Lambda)\left[\tan\beta_{2}+\frac{\sin\Lambda}{\lambda}\right]\end{array}\right\}$$
(29)

Inner section mechanism. The lengths of the velocity discontinuities of the partial mechanism of Fig. 20 are

$$\begin{split} L_{\mathrm{H'X'}} &= R(\beta_{2}^{\prime} + \Lambda^{\prime}) \\ L_{\mathrm{A'H'}} &= R(\beta_{1}^{\prime} - \beta_{2}^{\prime}) \\ L_{\mathrm{CA'}} &= L_{\mathrm{E'F'}} = R \frac{\cos(\beta_{1}^{\prime} + \Lambda^{\prime} - \omega)}{\sin(\beta_{1}^{\prime} - \omega)} \\ L_{\mathrm{CD'}} &= \int_{\beta_{2}^{\prime}}^{\beta_{1}^{\prime}} r d\beta \\ &= R(\beta_{1}^{\prime} - \beta_{2}^{\prime}) \left[ \frac{\cos(\beta_{1}^{\prime} + \Lambda^{\prime} - \omega)}{\sin(\beta_{1}^{\prime} - \omega)} + \sin \Lambda^{\prime} + \frac{\lambda^{\prime}}{2} (\beta_{1}^{\prime} - \beta_{2}^{\prime}) \right] \\ L_{\mathrm{A'G'}} &= R(\beta_{1}^{\prime} - \beta_{2}^{\prime}) \left[ \sin \Lambda^{\prime} + \frac{\lambda^{\prime}(\beta_{1}^{\prime} - \beta_{2}^{\prime})}{2} \right] \\ L_{\mathrm{G'F'}} &= R \left( \frac{\pi}{2} + \beta_{2}^{\prime} - \xi_{0} \right) [\lambda^{\prime}(\beta_{1}^{\prime} - \beta_{2}^{\prime} + \tan \beta_{2}^{\prime}) + \sin \Lambda^{\prime}] \\ L_{\mathrm{D'E'}} &= \left( \frac{\pi}{2} + \beta_{2}^{\prime} - \xi_{0} \right) (L_{\mathrm{CA'}} + L_{\mathrm{G'F'}}) \\ L_{\mathrm{F'I'}} &= \lambda^{\prime} R(\beta_{1}^{\prime} - \beta_{2}^{\prime}) \\ L_{\mathrm{I'X'}} &= \frac{\sin \xi_{0}}{\sin \xi_{1}} R(\lambda^{\prime} \tan \beta_{2}^{\prime} + \sin \Lambda^{\prime}) \\ L_{\mathrm{F'F'''}} &= L_{\mathrm{I'X'}} \sin \xi_{1} + \lambda^{\prime} R \sin \xi_{0} (\beta_{1}^{\prime} - \beta_{2}^{\prime}) \end{split}$$

$$\tag{30}$$

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The relative velocities for the partial mechanism are  $v_{\rm H'X'} = v_0 \frac{\cos\beta_2'}{\lambda'}$ 

$$v_{A'H'} = v_0 \frac{\cos \beta'}{\lambda'}$$

$$v_{CA'} = v_0 \cos \beta'_1$$

$$v_{CD'} = v_{D'E'} = v_0 \sin \beta'_1$$

$$v_{E'F'} = v_0 \sin \beta'_1 \tan \xi_0$$

$$v_{F'I'} = v_0 \frac{\sin(\beta' + \Lambda') \tan \xi_0}{\lambda'}$$

$$v_{I'X'} = v_0 \frac{d \cos \beta'_2 \tan \xi_1}{\lambda' R} (v_{I'X'} \text{ varies linearly along I'X'})$$

$$v_{I'X'} = \sum_{i=1}^{i} \left[ \sin(\beta' + \Lambda') - \frac{1}{\lambda' R} \right]$$

$$v_{I'X'} = \sum_{i=1}^{i} \left[ \sin(\beta' + \Lambda') - \frac{1}{\lambda' R} \right]$$

$$v_{\mathrm{F'F''}} = \frac{v_0}{\cos\xi_0} \left[ \frac{\sin(\beta' + \Lambda')}{\lambda'} - \sin\beta_1' \right]$$

The work done along the velocity discontinuities is

$$s_{u} \begin{cases} \alpha \left( L_{H'X'} v_{H'X'} + v_{0} R \frac{\sin \beta'_{1} - \sin \beta'_{2}}{\lambda'} \right) \\ + L_{CA'} v_{CA'} + L_{CD'} v_{CD'} + L_{D'E'} v_{D'E'} + L_{A'G'} v_{A'G'} \\ + L_{G'F'} v_{G'F'} + L_{E'F'} v_{E'F'} + v_{0} R \tan \xi_{0} [\cos(\beta'_{2} + \Lambda')] \\ - \cos(\beta'_{1} + \Lambda')] \\ + v_{0} R \tan \xi_{0} \frac{(\lambda' \tan \beta'_{2} + \sin \Lambda')^{2} \cos \beta'_{2}}{2\lambda'} + L_{F'F''} v_{F'F''} \end{cases}$$
(32)

The total work for the inner section of the mechanism also includes the work done within the five plastically deforming regions A'G'H', CD'E'F'G'A', G'F'I'H', F'I'I''F'' and I'X'I'', as follows.

# Region A'G'H'

The derivation is the same as for the inner section of the mechanism.

$$v_{0}s_{u}R\left\{\begin{array}{l}-2\sin(\beta_{2}'+\Lambda')+\cos(\beta_{2}'+\Lambda')\frac{\sin\Lambda'}{\lambda'}\\-\cos(\beta_{1}'+\Lambda')\left[2(\beta_{1}'-\beta_{2}')+\frac{\sin\Lambda'}{\lambda}\right]\\-\frac{1}{2}\sin(\beta_{1}'+\Lambda')\\\left[(\beta_{1}'-\beta_{2}')^{2}+2(\beta_{1}'-\beta_{2}')\frac{\sin\Lambda'}{\lambda'}-4\right]\end{array}\right\}$$

$$(33)$$

Region CD'E'F'G'A'

$$v_0 s_u R \sin\beta_1' \int_0^{\beta_1' + \pi/2 - \xi_0} \frac{\cos(\beta_1' + \Lambda' - \omega)}{\sin(\beta_1' - \omega)} \,\mathrm{d}\psi$$

Therefore

$$v_0 s_u R \cos(\beta_1' + \Lambda' - \omega) \frac{\sin\beta_1'}{\sin(\beta_1' - \omega)} \left(\frac{\pi}{2} + \beta_1' - \xi_0\right) \tag{34}$$

Region G'F'I'H'

$$v_{0}s_{u}R\left(\frac{\pi}{2}+\beta'_{2}-\xi_{0}\right)\left[\begin{array}{c}2[\cos(\beta'_{2}+\Lambda')-\cos(\beta'_{1}+\Lambda')]\\-\sin(\beta'_{1}+\Lambda')\left(\beta'_{1}-\beta'_{2}+\tan\beta'_{2}+\frac{\sin\Lambda'}{\lambda'}\right)\\+\sin(\beta'_{2}+\Lambda')\left(\tan\beta'_{2}+\frac{\sin\Lambda'}{\lambda'}\right)\end{array}\right]$$
(35)

Region F'I'I"F"

Velocities in this region are parallel to the *y*-axis, and are given by equation (9) as a function of angle  $\beta'$ . Equation (9) can be expressed in terms of the *x* coordinate as

|← s/2 Fig. 20. Inner upper-bound mechanism

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NOTATION

$$v' = \frac{v_0}{\lambda' \cos \xi_0} \sin \left( \frac{\beta_2' - \tan \beta_2' + \arccos \lambda' - \frac{\sin \Lambda'}{\lambda'}}{+\frac{s/2 - x}{\lambda' R \cos \xi_0} - \frac{1}{\cos \beta_2' \cos \xi_0}} \right)$$

The shear strain rate is

$$\dot{\gamma} = \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}$$
$$= -\frac{v_0}{\lambda'^2 R \cos^2 \xi_0} \cos \left( \begin{array}{c} \beta'_2 - \tan \beta'_2 + \Lambda' - \frac{\sin \Lambda'}{\lambda'} \\ + \frac{s/2 - x}{\lambda' R \cos \xi_0} - \frac{1}{\cos \beta'_2 \cos \xi_0} \end{array} \right)$$

The work done in the region is

$$-\frac{s_{\mathrm{u}}v_0}{{\lambda'}^2R\cos^2\xi_0}\int_{x_1}^{x_2}\int_0^y\cos\left(\frac{\beta_2'-\tan\beta_2'+\Lambda'-\frac{\sin\Lambda'}{\lambda'}}{+\frac{s/2-x}{\lambda'R\cos\xi_0}-\frac{1}{\cos\beta_2'\cos\xi_0}}\right)\mathrm{d}y\,\mathrm{d}x$$

where

$$x_{1} = \frac{s}{2} - \frac{R}{\cos\beta'_{2}} [\lambda' + \cos\xi_{0}\sin(\beta'_{2} - \Lambda')]$$

$$x_{2} = \frac{s}{2} - \frac{R}{\cos\beta'_{2}} [\lambda' + \cos\xi_{0}\sin(\beta'_{2} - \Lambda')]$$

$$-\lambda'R\cos\xi_{0}(\beta'_{1} - \beta'_{2})$$

$$y = \left(\frac{s}{2} - \frac{\lambda'R}{\cos\beta'_{2}} - x\right) \tan\xi_{0}$$

and therefore the work in F'I'I"F" is

$$s_{u}v_{0}\tan\xi_{0}R\left\{\begin{array}{c}\cos(\beta_{1}^{\prime}+\Lambda^{\prime})-\cos(\beta_{2}^{\prime}+\Lambda^{\prime})\\+\sin(\beta_{1}^{\prime}+\Lambda^{\prime})(\beta_{1}^{\prime}-\beta_{2}^{\prime})\\\left[\sin(\beta_{1}^{\prime}+\Lambda^{\prime})-\sin(\beta_{2}^{\prime}+\Lambda^{\prime})\right]\frac{\sin(\beta_{2}^{\prime}+\Lambda^{\prime})}{\lambda^{\prime}\cos\beta_{2}^{\prime}}\right\}$$

Region I'XI"

The shear strain rate is

$$\dot{\gamma} = \frac{\partial}{\partial x} \left\{ -v_0 \left[ \frac{\cos \beta_2'}{\lambda' R} \left( R + \frac{s/2 - \lambda' / \cos \beta_2' - x}{\cos^2 \xi_1} \right) - 1 \right] \right\}$$
$$= v_0 \frac{\cos \beta_2'}{\lambda' R \cos^2 \xi_1}$$

The work done in the region is

$$s_{\rm u} \int_{A} \dot{\gamma} dA = s_{\rm u} \dot{\gamma} A = s_{\rm u} v_0 \frac{\cos \beta_2' \tan \xi_1}{2\lambda' R}$$
(37)

#### Kinematic mechanism B

The total work done in one quarter of mechanism B (Fig. 11) can be calculated using, a similar procedure as for the outer part of mechanism A, making the following substitutions:  $\lambda = \cos(\arccos \alpha)$ ,  $\omega = 0$ ,  $\beta_1 = \pi/4$  and  $\beta_2 = \beta$ .

The geometrical parameters, velocities and work calculations are given by equations (23)–(29), with the exception of lengths  $L_{CA}$  and  $L_{CD}$ 

$$L_{CA} = R\sqrt{2} \left[ \sin\left(\frac{\pi}{4} - \frac{\arccos \alpha}{2}\right) + \frac{s}{2R} \right]$$
$$L_{CD} = R\left(\frac{\pi}{4} - \beta\right) \left[ \sqrt{2}\frac{s}{2R} + \cos\left(\frac{\arccos \alpha}{2}\right) \left(1 + \frac{\pi}{8} - \frac{\beta}{2}\right) \right]$$

and the work done in section CDEFGA

$$v_0 s_u R \frac{\sqrt{2}}{2} \int_0^{5\pi/4} \left[ \sin\left(\frac{\pi}{4} - \frac{\arccos \alpha}{2}\right) + \frac{s}{2R} \right] d\chi$$

$$= v_0 s_u R \left[ \sin\left(\frac{\pi}{4} - \frac{\arccos \alpha}{2}\right) + \frac{s}{2R} \right] \frac{3\sqrt{2}\pi}{8}$$
(38)

matrix of equality constraint coefficients
pile diameter
distance from X' along discontinuity X'I'
modulus of elasticity of pile
undrained modulus of elasticity
yield function for stress set i
length of discontinuity i
number of stress nodes
lateral bearing capacity factor
peak lateral bearing capacity factor
lateral bearing capacity factor for
equivalent non-circular pile
lateral bearing capacity factor for $s = 0$
single pile lateral bearing capacity factor
ultimate lateral load per unit length
vectors of prescribed and optimisable
forces respectively
pile radius
cylindrical coordinates
centre-to-centre pile spacing
pile spacing corresponding to $N_{\rm pp}$
undrained shear strength
pile spacing beyond which $N_{\rm p} = N_{\rm p1}$
angles specifying radial characteristics
pile lateral velocity
internal lateral velocities of failure
mechanism
relative velocity of discontinuity i
work done by internal stresses
lateral pile displacement
adhesion factor
angle specifying tangential characteristics
geometrical optimisation parameters
shear strain rate
$\arccos \lambda$
Poisson's ratio of pile
undrained Poisson's ratio

- $\xi_0, \xi_1$  angles defining the kinematic mechanism  $\sigma$  vector of stress variables
  - $\tau_{\rm f}$   $\,$  ultimate shear stress along discontinuity

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 $\beta_1, \beta_2, \beta'_1, \beta'_2,$ 

(36)

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